

Markov L_2 inequality with the Gegenbauer weight

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Abstract

For the Gegenbauer weight function $w_\lambda(t) = (1 - t^2)^{\lambda-1/2}$, $\lambda > -1/2$, we denote by $\|\cdot\|_{w_\lambda}$ the associated L_2 -norm,

$$\|f\|_{w_\lambda} := \left(\int_{-1}^1 w_\lambda(t) f^2(t) dt \right)^{1/2}.$$

We study the Markov inequality

$$\|p'\|_{w_\lambda} \leq c_n(\lambda) \|p\|_{w_\lambda}, \quad p \in \mathcal{P}_n,$$

where \mathcal{P}_n is the class of algebraic polynomials of degree not exceeding n . Upper and lower bounds for the best Markov constant $c_n(\lambda)$ are obtained, which are valid for all $n \in \mathbb{N}$ and $\lambda > -\frac{1}{2}$.

1 Introduction and statement of the results

Throughout this paper \mathcal{P}_n stands for the class of algebraic polynomials of degree not exceeding n .

For the Gegenbauer weight function $w_\lambda(t) = (1 - t^2)^{\lambda-1/2}$, $\lambda > -1/2$, we denote by $\|\cdot\|_{w_\lambda}$ the associated L_2 -norm,

$$\|f\|_{w_\lambda} := \left(\int_{-1}^1 w_\lambda(t) f^2(t) dt \right)^{1/2}.$$

Here we study the Markov inequality in this norm for the first derivative of polynomials from \mathcal{P}_n , in particular, we are interested in the best Markov constant

$$c_n(\lambda) = \sup_{\substack{p \in \mathcal{P}_n \\ p \neq 0}} \frac{\|p'\|_{w_\lambda}}{\|p\|_{w_\lambda}}.$$

Let us start with a brief account of the known results.

In the case $\lambda = \frac{1}{2}$ (the case of a constant weight function), E. Schmidt proved that

$$c_n(1/2) = \frac{(2n+3)^2}{4\pi} \left(1 - \frac{\pi^2 - 3}{3(2n+3)^2} + \frac{16R}{(2n+3)^4} \right)^{-1}, \quad -6 < R < 13.$$

Nikolov [3] studied two other particular cases, $\lambda = 0, 1$, and proved the following two-sided estimates for the corresponding Markov constants:

$$\begin{aligned} 0.472135n^2 \leq c_n(0) \leq 0.478849(n+2)^2, \\ 0.248549n^2 \leq c_n(1) \leq 0.256861(n + \frac{5}{2})^2. \end{aligned} \tag{1.1}$$

In [1] we obtained an upper bound for $c_n(\lambda)$, which is valid for all n and λ :

$$c_n(\lambda) \leq \frac{(n+1)(n+2\lambda+1)}{2\sqrt{2\lambda+1}}.$$

This result has been improved in the recent paper [5], where the following theorem was proved:

Theorem A For all $\lambda > -\frac{1}{2}$ and $n \geq 3$, the best constant $c_n(\lambda)$ in the Markov inequality

$$\|p'_n\|_{w_\lambda} \leq c_n(\lambda) \|p_n\|_{w_\lambda}, \quad p_n \in \mathcal{P}_n,$$

admits the estimates

$$\frac{1}{4} \frac{n^2(n+\lambda)^2}{(\lambda+1)(\lambda+2)} < [c_n(\lambda)]^2 < \frac{n(n+2\lambda+2)^3}{(\lambda+2)(\lambda+3)}, \quad \lambda \geq 2; \quad (1.2)$$

$$\frac{(n+\lambda)^2(n+2\lambda')^2}{(2\lambda+1)(2\lambda+5)} < [c_n(\lambda)]^2 < \frac{(n+\lambda+\lambda''+2)^4}{2(2\lambda+1)\sqrt{2\lambda+5}}, \quad \lambda > -\frac{1}{2}, \quad (1.3)$$

where $\lambda' = \min\{0, \lambda\}$, $\lambda'' = \max\{0, \lambda\}$.

It has been also proved in [5] that

$$[c_n(\lambda)]^2 \asymp \frac{1}{\lambda^2} n(n+2\lambda)^3,$$

which shows that the upper bound in (1.2) has the right order in both n and λ . The lower bound in (1.2) is inferior to the one in (1.3), it appears in (1.2) just to indicate that, roughly, for a fixed λ and large n the sharp Markov constant is identified within a factor not exceeding two. Although the upper bound in (1.3) is not of the right order with respect to λ , for moderate λ (say, $\lambda \leq 25$) it is superior to the one in (1.2).

In the present paper we prove two-sided estimates for $c_n(\lambda)$, valid for all $\lambda > -1/2$, which are of the same nature as (and slightly sharper than) those in (1.3). The approaches for their derivation however are different. In [5], the results are obtained through estimation of appropriate matrix norms. Here, we identify the reciprocal of the squared best Markov constant as the smallest zero of a related orthogonal polynomial, then exploit the associated three-term recurrence relation to evaluate its lower degree coefficients and eventually derive estimates for its smallest zero. Let us mention that a similar relation between the best constant in the L_2 Markov inequality with the Laguerre weight function and the smallest zero of an orthogonal polynomial is given in [2, p. 85], and in [4] we applied a similar approach to obtain bounds for the best Markov constant in the Laguerre case.

Our main result is the following theorem:

Theorem 1.1 For all $n \geq 3$ and for every $\lambda > -\frac{1}{2}$, the best constant $c_n(\lambda)$ in the Markov inequality

$$\|p'\|_{w_\lambda} \leq c_n(\lambda) \|p\|_{w_\lambda}, \quad p \in \mathcal{P}_n, \quad (1.4)$$

admits the estimates

$$\frac{(n+1)(n+\lambda+\frac{1}{2})^2(n+2\lambda)}{(2\lambda+1)(2\lambda+5)} \leq c_n^2(\lambda) \leq \frac{(n+\frac{5}{4}\lambda+\frac{9}{8})^4}{2(2\lambda+1)\sqrt{2\lambda+5}}. \quad (1.5)$$

By setting $\lambda = 0, 1$ in (1.5), we obtain an improvement of the upper bounds in (1.1), and combination with the lower bounds in (1.1) yields rather tight estimates.

Corollary 1.2 For the Chebyshev weights $w_0(x) = \frac{1}{\sqrt{1-x^2}}$ and $w_1(x) = \sqrt{1-x^2}$, we have

$$\begin{aligned} 0.472135 n^2 &\leq c_n(0) \leq 0.472871 \left(n + \frac{9}{8}\right)^2, \\ 0.248549 n^2 &\leq c_n(1) \leq 0.250987 \left(n + \frac{19}{8}\right)^2. \end{aligned}$$

For the proof of Theorem 1.1 we obtain separately estimates for $c_n(\lambda)$ in the cases of even and odd n (Theorems 4.2 and 4.4). These estimates are slightly sharper than the ones in Theorem 1.1, in particular, they yield the following asymptotic inequalities:

Corollary 1.3 *For every $n \geq 3$, there holds*

$$\frac{(n+2)(n-1)n^2}{4} \leq \lim_{\lambda \rightarrow -\frac{1}{2}} (2\lambda+1) c_n^2(\lambda) \leq \frac{n^2(n+1)^2}{4}. \quad (1.6)$$

The paper is organised as follows. In Sect. 2 we show that the reciprocal of the squared best Markov constant, $1/[c_n(\lambda)]^2$, is equal to the smallest zero of an orthogonal polynomial of degree $m = \lfloor \frac{n+1}{2} \rfloor$ (different in the cases $n = 2m$ and $n = 2m - 1$), and we derive the three-term recurrence relation satisfied by these orthogonal polynomials. Based on the three-term recurrence relations, in Sect. 3 we evaluate and estimate the lowest degree coefficients of the m -th orthogonal polynomial. In Sect. 4 we prove estimates for $c_n(\lambda)$ in the cases of even and odd n (Theorems 4.2 and 4.4), and derive as consequences Theorem 1.1 and Corollary 1.3.

2 $c_n^2(\lambda)$ and the extreme zero of an orthogonal polynomial

In a recent paper [1] we showed that the extreme polynomial in the Markov inequality (1.4) is even or odd if n is even or odd. The following theorem summarizes some of the results obtained in [1]:

Theorem 2.1 *The best constant $c_n(\lambda)$ in the Markov inequality (1.4) is given by*

$$c_n(\lambda) = \begin{cases} 2\sqrt{\nu_m}, & n = 2m, \\ 2\sqrt{\tilde{\nu}_m}, & n = 2m - 1, \end{cases} \quad (2.1)$$

where ν_m and $\tilde{\nu}_m$ are the largest eigenvalues of the $m \times m$ positive definite matrices $\mathbf{C}_m^\top \mathbf{C}_m$ and $\tilde{\mathbf{C}}_m^\top \tilde{\mathbf{C}}_m$, respectively, given by

$$\mathbf{C}_m = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \cdots & \alpha_1\beta_m \\ 0 & \alpha_2\beta_2 & \cdots & \alpha_2\beta_m \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_m\beta_m \end{pmatrix}, \quad \tilde{\mathbf{C}}_m = \begin{pmatrix} \tilde{\alpha}_1\tilde{\beta}_1 & \tilde{\alpha}_1\tilde{\beta}_2 & \cdots & \tilde{\alpha}_1\tilde{\beta}_m \\ 0 & \tilde{\alpha}_2\tilde{\beta}_2 & \cdots & \tilde{\alpha}_2\tilde{\beta}_m \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\alpha}_m\tilde{\beta}_m \end{pmatrix}. \quad (2.2)$$

Here,

$$\alpha_k := (2k - 1 + \lambda)h_{2k-1}, \quad \beta_k := \frac{1}{h_{2k}}; \quad (2.3)$$

$$\tilde{\alpha}_k := (2k - 2 + \lambda)h_{2k-2}, \quad \tilde{\beta}_k := \frac{1}{h_{2k-1}}, \quad (2.4)$$

with

$$h_i^2 := h_{i,\lambda}^2 := \frac{\Gamma(i+2\lambda)}{(i+\lambda)\Gamma(i+1)}. \quad (2.5)$$

Clearly, matrices \mathbf{C}_m and $\tilde{\mathbf{C}}_m$ can be represented as

$$\begin{aligned} \mathbf{C}_m &= \text{diag}(\alpha_1, \dots, \alpha_m) \mathbf{T}_m \text{diag}(\beta_1, \dots, \beta_m), \\ \tilde{\mathbf{C}}_m &= \text{diag}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_m) \mathbf{T}_m \text{diag}(\tilde{\beta}_1, \dots, \tilde{\beta}_m), \end{aligned} \quad (2.6)$$

where \mathbf{T}_m is an upper tri-diagonal $m \times m$ matrix with non-zero entries equal to 1,

$$\mathbf{T}_m = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Since $\mathbf{C}_m^\top \mathbf{C}_m \sim \mathbf{C}_m \mathbf{C}_m^\top$ and $\tilde{\mathbf{C}}_m^\top \tilde{\mathbf{C}}_m \sim \tilde{\mathbf{C}}_m \tilde{\mathbf{C}}_m^\top$, we conclude that

$$\begin{aligned} \nu_m & \text{ is the largest eigenvalue of the matrix } \mathbf{A}_m := \mathbf{C}_m \mathbf{C}_m^\top, \\ \tilde{\nu}_m & \text{ is the largest eigenvalue of the matrix } \tilde{\mathbf{A}}_m := \tilde{\mathbf{C}}_m \tilde{\mathbf{C}}_m^\top. \end{aligned} \quad (2.7)$$

It turns out that it is advantageous to work with the inverse matrices $\mathbf{B}_m := \mathbf{A}_m^{-1}$ and $\tilde{\mathbf{B}}_m := \tilde{\mathbf{A}}_m^{-1}$, respectively, as \mathbf{B}_m and $\tilde{\mathbf{B}}_m$ are tri-diagonal matrices. Below we demonstrate this for \mathbf{B}_m .

The matrix \mathbf{T}_m^{-1} is two-diagonal, namely,

$$\mathbf{T}_m^{-1} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \quad (2.8)$$

For $\mathbf{B}_m = \mathbf{A}_m^{-1} = (\mathbf{C}_m^\top)^{-1} \mathbf{C}_m^{-1} = (\mathbf{C}_m^{-1})^\top \mathbf{C}_m^{-1}$, using (2.6), we have

$$\begin{aligned} \mathbf{B}_m &= (\text{diag}(\beta_1^{-1}, \dots, \beta_m^{-1}) \mathbf{T}_m^{-1} \text{diag}(\alpha_1^{-1}, \dots, \alpha_m^{-1}))^\top \text{diag}(\beta_1^{-1}, \dots, \beta_m^{-1}) \mathbf{T}_m^{-1} \text{diag}(\alpha_1^{-1}, \dots, \alpha_m^{-1}) \\ &= \text{diag}(\alpha_1^{-1}, \dots, \alpha_m^{-1}) (\mathbf{T}_m^{-1})^\top \text{diag}(\beta_1^{-2}, \dots, \beta_m^{-2}) \mathbf{T}_m^{-1} \text{diag}(\alpha_1^{-1}, \dots, \alpha_m^{-1}). \end{aligned}$$

Making use of (2.8), we perform the multiplications to conclude that, indeed, \mathbf{B}_m is tri-diagonal. We formulate the result below:

Proposition 2.2 *The matrix $\mathbf{A}_m^{-1} =: \mathbf{B}_m = (b_{i,j})_{m \times m}$ is symmetric and tri-diagonal, with elements*

$$b_{1,1} = \frac{1}{\alpha_1^2 \beta_1^2}, \quad (2.9)$$

$$b_{k,k} = \frac{1}{\alpha_k^2} \left(\frac{1}{\beta_{k-1}^2} + \frac{1}{\beta_k^2} \right), \quad k = 2, \dots, m, \quad (2.10)$$

$$b_{k,k+1} = -\frac{1}{\alpha_k \alpha_{k+1} \beta_k^2}, \quad k = 1, \dots, m-1. \quad (2.11)$$

The same conclusion applies to the matrix $\tilde{\mathbf{A}}_m^{-1} =: \tilde{\mathbf{B}}_m = (\tilde{b}_{i,j})_{m \times m}$, with the b 's, α 's and β 's replaced by the \tilde{b} 's, $\tilde{\alpha}$'s and $\tilde{\beta}$'s.

Thus, \mathbf{B}_m and $\tilde{\mathbf{B}}_m$ are Jacobi matrices, which are positive definite as inverse of the positive definite matrices \mathbf{A}_m and $\tilde{\mathbf{A}}_m$. The characteristic polynomials of \mathbf{B}_m and $\tilde{\mathbf{B}}_m$,

$$P_m(\mu) = \det(\mu \mathbf{E}_m - \mathbf{B}_m), \quad \tilde{P}_m(\mu) = \det(\mu \mathbf{E}_m - \tilde{\mathbf{B}}_m),$$

are determined by three-term recurrence relations, and, by Favard's theorem, $\{P_m\}$ and $\{\tilde{P}_m\}$ constitute two sequences of orthogonal polynomials with respect to measures supported on the positive axis. Let $\mu_1 < \mu_2 < \dots < \mu_m$ and $\tilde{\mu}_1 < \tilde{\mu}_2 < \dots < \tilde{\mu}_m$ be the zeros of P_m and \tilde{P}_m , respectively, i.e., the eigenvalues of \mathbf{B}_m and $\tilde{\mathbf{B}}_m$. Since the latter are reciprocal to the eigenvalues of \mathbf{A}_m and $\tilde{\mathbf{A}}_m$, in particular, $\nu_m = \mu_1^{-1}$ and $\tilde{\nu}_m = \tilde{\mu}_1^{-1}$, Theorem 2.1, (2.7) and Proposition 2.2 yield the following

Theorem 2.3 *The best constant $c_n(\lambda)$ in the Markov inequality (1.4) is given by*

$$c_n(\lambda) = \begin{cases} \frac{2}{\sqrt{\mu_1}}, & n = 2m, \\ \frac{2}{\sqrt{\tilde{\mu}_1}}, & n = 2m-1, \end{cases} \quad (2.12)$$

where μ_1 and $\tilde{\mu}_1$ are the smallest zeros of monic polynomials P_m and \tilde{P}_m , orthogonal with respect to a measure supported on \mathbb{R}_+ . The polynomials $\{P_k\}$ are defined by the three-term recurrence relation

$$\begin{aligned} P_k(\mu) &= \left[\mu - \frac{1}{\alpha_k^2} \left(\frac{1}{\beta_{k-1}^2} + \frac{1}{\beta_k^2} \right) \right] P_{k-1}(\mu) - \frac{1}{\alpha_{k-1}^2 \alpha_k^2 \beta_{k-1}^4} P_{k-2}(\mu), \quad k \geq 2, \\ P_0(\mu) &= 1, \quad P_1(\mu) = \mu - \frac{1}{\alpha_1^2 \beta_1^2}. \end{aligned} \quad (2.13)$$

The polynomials $\{\tilde{P}_k\}$ satisfy the same recurrence relation, with the α 's and β 's replaced by the $\tilde{\alpha}$'s and $\tilde{\beta}$'s.

We renormalise polynomials $\{P_k\}_0^m$ and $\{\tilde{P}_k\}_0^m$ by setting $Q_0 = P_0 = \tilde{Q}_0 = \tilde{P}_0 = 1$ and

$$Q_k = d_k P_k, \quad \tilde{Q}_k = \tilde{d}_k \tilde{P}_k, \quad k = 1, \dots, m$$

so that

$$Q_k(0) = \tilde{Q}_k(0) = 1, \quad k = 0, \dots, m \quad (2.14)$$

(note that this is possible because all the zeros of P_k and \tilde{P}_k are positive).

For $k = 1, \dots, m$, we have $P_k(\mu) = \det(\mu \mathbf{E}_k - \mathbf{B}_k)$ and $\tilde{P}_k(\mu) = \det(\mu \mathbf{E}_k - \tilde{\mathbf{B}}_k)$, therefore,

$$\begin{aligned} P_k(0) &= \det(-\mathbf{B}_k) = (-1)^k \det(\mathbf{B}_k) = (-1)^k \det(\mathbf{A}_k^{-1}) = (-1)^k \det(\mathbf{A}_k)^{-1}, \\ \tilde{P}_k(0) &= \det(-\tilde{\mathbf{B}}_k) = (-1)^k \det(\tilde{\mathbf{B}}_k) = (-1)^k \det(\tilde{\mathbf{A}}_k^{-1}) = (-1)^k \det(\tilde{\mathbf{A}}_k)^{-1}. \end{aligned}$$

Since $\mathbf{A}_k = \mathbf{C}_k \mathbf{C}_k^\top$ and $\tilde{\mathbf{A}}_k = \tilde{\mathbf{C}}_k \tilde{\mathbf{C}}_k^\top$, we make use of (2.6) (with m replaced by k) to obtain

$$\det(\mathbf{A}_k) = \det(\mathbf{C}_k)^2 = \prod_{i=1}^k \alpha_i^2 \beta_i^2, \quad \det(\tilde{\mathbf{A}}_k) = \det(\tilde{\mathbf{C}}_k)^2 = \prod_{i=1}^k \tilde{\alpha}_i^2 \tilde{\beta}_i^2.$$

Consequently,

$$P_k(0) = \frac{(-1)^k}{\prod_{i=1}^k \alpha_i^2 \beta_i^2}, \quad \tilde{P}_k(0) = \frac{(-1)^k}{\prod_{i=1}^k \tilde{\alpha}_i^2 \tilde{\beta}_i^2} \Rightarrow d_k = (-1)^k \prod_{i=1}^k \alpha_i^2 \beta_i^2, \quad \tilde{d}_k = (-1)^k \prod_{i=1}^k \tilde{\alpha}_i^2 \tilde{\beta}_i^2.$$

Thus, the renormalised to satisfy (2.14) polynomials $\{Q_k\}$ and $\{\tilde{Q}_k\}$ are given by

$$Q_k(\mu) = (-1)^k \left(\prod_{i=1}^k \alpha_i^2 \beta_i^2 \right) P_k(\mu), \quad \tilde{Q}_k(\mu) = (-1)^k \left(\prod_{i=1}^k \tilde{\alpha}_i^2 \tilde{\beta}_i^2 \right) \tilde{P}_k(\mu), \quad k = 1, \dots, m. \quad (2.15)$$

From (2.13) it is easy to deduce the recurrence relations satisfied by $\{Q_k\}$ and $\{\tilde{Q}_k\}$.

Proposition 2.4 *The polynomials $\{Q_k\}$ in (2.15) satisfy the recurrence relation*

$$\begin{aligned} Q_k(\mu) - Q_{k-1}(\mu) &= \frac{\beta_k^2}{\beta_{k-1}^2} [Q_{k-1}(\mu) - Q_{k-2}(\mu)] - \alpha_k^2 \beta_k^2 \mu Q_{k-1}(\mu), \quad k \geq 2, \\ Q_0(\mu) &= 1, \quad Q_1(\mu) = 1 - \alpha_1^2 \beta_1^2 \mu. \end{aligned} \quad (2.16)$$

The polynomials $\{\tilde{Q}_k\}$ in (2.15) satisfy the same recurrence relation, with the α 's and β 's replaced by the $\tilde{\alpha}$'s and $\tilde{\beta}$'s.

3 The lowest degree coefficients of Q_m and \tilde{Q}_m

In view of (2.14), we may write polynomials Q_k and \tilde{Q}_k , $k \geq 1$, in the form

$$\begin{aligned} Q_k(\mu) &= 1 - A_{1,k} \mu + A_{2,k} \mu^2 - \cdots + (-1)^k A_{k,k} \mu^k, \\ \tilde{Q}_k(\mu) &= 1 - \tilde{A}_{1,k} \mu + \tilde{A}_{2,k} \mu^2 + \cdots + (-1)^k \tilde{A}_{k,k} \mu^k. \end{aligned} \quad (3.1)$$

Our goal now is to find expressions for $A_{i,m}$, $\tilde{A}_{i,m}$, $i = 1, 2$. First of all, we make use of (2.3)–(2.4) to find the explicit form of the coefficients occurring in recurrence formulae for Q_k and \tilde{Q}_k . We have

$$\frac{\beta_k^2}{\beta_{k-1}^2} = \frac{k(2k-1)(2k+\lambda)}{(k-1+\lambda)(2k-2+\lambda)(2k-1+2\lambda)}, \quad \alpha_k^2 \beta_k^2 = \frac{2k(2k-1+\lambda)(2k+\lambda)}{2k-1+2\lambda}, \quad (3.2)$$

$$\frac{\tilde{\beta}_k^2}{\tilde{\beta}_{k-1}^2} = \frac{(k-1)(2k-1)(2k-1+\lambda)}{(k-1+\lambda)(2k-3+\lambda)(2k-3+2\lambda)}, \quad \tilde{\alpha}_k^2 \tilde{\beta}_k^2 = \frac{(2k-1)(2k-2+\lambda)(2k-1+\lambda)}{2(k-1+\lambda)}. \quad (3.3)$$

By substituting these quantities in the recurrence formulae in Proposition 2.4 and replacing k by m , we obtain

$$\begin{aligned} Q_m(\mu) - Q_{m-1}(\mu) &= \frac{m(2m-1)(2m+\lambda)}{(m-1+\lambda)(2m-2+\lambda)(2m-1+2\lambda)} [Q_{m-1}(\mu) - Q_{m-2}(\mu)] \\ &\quad - \frac{2m(2m-1+\lambda)(2m+\lambda)}{2m-1+2\lambda} \mu Q_{m-1}(\mu), \end{aligned} \quad (3.4)$$

$$\begin{aligned} \tilde{Q}_m(\mu) - \tilde{Q}_{m-1}(\mu) &= \frac{(m-1)(2m-1)(2m-1+\lambda)}{(m-1+\lambda)(2m-3+\lambda)(2m-3+2\lambda)} [\tilde{Q}_{m-1}(\mu) - \tilde{Q}_{m-2}(\mu)] \\ &\quad - \frac{(2m-1)(2m-2+\lambda)(2m-1+\lambda)}{2(m-1+\lambda)} \mu \tilde{Q}_{m-1}(\mu). \end{aligned} \quad (3.5)$$

Lemma 3.1 *For every $m \in \mathbb{N}_0$ there holds*

$$(i) \quad A_{1,m} = \frac{m(m+1)(m+\lambda)(m+\lambda+1)}{2\lambda+1}; \quad (ii) \quad \tilde{A}_{1,m} = \frac{m(m+\lambda)(m^2+\lambda m - \frac{1}{2})}{2\lambda+1}.$$

Proof. (i) The formula is true for $m = 0$, since $Q_0(\mu) = 1$, and hence $A_{1,0} = 0$. Clearly, (i) holds for $m = 1$, too, since, by (2.16) and (3.2),

$$A_{1,1} = \alpha_1^2 \beta_1^2 = \frac{2(\lambda+1)(\lambda+2)}{2\lambda+1}.$$

We set $D_{1,k} := A_{1,k} - A_{1,k-1}$, $k \in \mathbb{N}$, then claim (i) is equivalent to

$$D_{1,m} = \frac{2m(m+\lambda)(2m+\lambda)}{2\lambda+1}, \quad m \in \mathbb{N}, \quad (3.6)$$

and it is true for $m = 1$, since $D_{1,1} = A_{1,1}$. We shall prove (3.6) by induction with respect to m . To this end, we differentiate (3.4) in μ and then set $\mu = 0$, making use of (3.1), to obtain the recurrence formula

$$D_{1,m} = \frac{m(2m-1)(2m+\lambda)}{(m-1+\lambda)(2m-2+\lambda)(2m-1+2\lambda)} D_{1,m-1} + \frac{2m(2m-1+\lambda)(2m+\lambda)}{2m-1+2\lambda}.$$

Assuming that (3.6) is true for $m-1$, $m \geq 2$, we substitute the expression for D_{m-1} in the above formula to verify that (3.6) holds for m :

$$\begin{aligned} D_{1,m} &= \frac{m(2m-1)(2m+\lambda)2(m-1)(m-1+\lambda)(2m-2+\lambda)}{(m-1+\lambda)(2m-2+\lambda)(2m-1+2\lambda)(2\lambda+1)} + \frac{2m(2m-1+\lambda)(2m+\lambda)}{2m-1+2\lambda} \\ &= \frac{2m(m+\lambda)(2m+\lambda)}{2\lambda+1}. \end{aligned}$$

(ii) Clearly, (ii) holds for $m = 0$, since $\tilde{A}_{1,0} = 0$, and it is also true for $m = 1$, since, by Proposition 2.4 and (3.3),

$$\tilde{A}_{1,1} = \tilde{\alpha}_1^2 \tilde{\beta}_1^2 = \frac{\lambda + 1}{2}.$$

Similarly to the proof of (i), we set $\tilde{D}_{1,k} = \tilde{A}_{1,k} - \tilde{A}_{1,k-1}$, $k \in \mathbb{N}$, then (ii) is equivalent to

$$\tilde{D}_{1,m} = \frac{(2m-1)(2m-1+\lambda)(2m-1+2\lambda)}{2(2\lambda+1)}, \quad (3.7)$$

and the latter is true for $m = 1$, since $\tilde{D}_{1,1} = \tilde{A}_{1,1}$. Similarly to the proof of (i), we obtain a recurrence relation by differentiating (3.5) and then substituting $\mu = 0$:

$$\tilde{D}_{1,m} = \frac{(m-1)(2m-1)(2m-1+\lambda)}{(m-1+\lambda)(2m-3+\lambda)(2m-3+2\lambda)} \tilde{D}_{1,m-1} + \frac{(2m-1)(2m-2+\lambda)(2m-1+\lambda)}{2(m-1+\lambda)}.$$

We observe that the right-hand side of (3.7) is obtained from the right-hand side of (3.6) by the change $m \mapsto m - 1/2$, and the same change transforms the recurrence relation for D_m into the recurrence relation for \tilde{D}_m . Therefore, (3.7) is a consequence of (3.6). \square

Remark 3.2 The coefficients $A_{1,m}$ and \tilde{A}_m are in fact the traces of matrices \mathbf{A}_m and $\tilde{\mathbf{A}}_m$, respectively, and they were evaluated in [1, Lemma 2.3]. We incorporate an alternative proof first, for the sake of completeness and, second, because the same approach is applied below for the evaluation of coefficients $A_{2,m}$ and $\tilde{A}_{2,m}$.

Next, we proceed with the evaluation of the coefficients $A_{2,m}$ and $\tilde{A}_{2,m}$. Let us set

$$D_{2,1} = 0, \quad D_{2,m} := A_{2,m} - A_{2,m-1}, \quad m \geq 2, \quad (3.8)$$

$$\tilde{D}_{2,1} = 0, \quad \tilde{D}_{2,m} := \tilde{A}_{2,m} - \tilde{A}_{2,m-1}, \quad m \geq 2. \quad (3.9)$$

Lemma 3.3 (i) The sequence $\{D_{2,m}\}$ defined by (3.8) satisfies the recurrence relation

$$D_{2,m} = \frac{m(2m-1)(2m+\lambda)}{(m-1+\lambda)(2m-2+\lambda)(2m-1+2\lambda)} D_{2,m-1} + \frac{2(m-1)m^2(m-1+\lambda)(m+\lambda)(2m-1+\lambda)(2m+\lambda)}{(2\lambda+1)(2m-1+2\lambda)}. \quad (3.10)$$

The solution of (3.10) with the initial condition $D_{2,1} = 0$ is given by

$$D_{2,m} = \frac{2(m-1)m(m+\lambda)(m+\lambda+1)(2m+\lambda)[m^2 + \lambda m - \frac{2}{2\lambda+3}]}{(2\lambda+1)(2\lambda+5)}. \quad (3.11)$$

(ii) The sequence $\{\tilde{D}_{2,m}\}$ defined by (3.9) satisfies the recurrence relation

$$\tilde{D}_{2,m} = \frac{(m-1)(2m-1)(2m-1+\lambda)}{(m-1+\lambda)(2m-3+\lambda)(2m-3+2\lambda)} \tilde{D}_{2,m-1} + \frac{(m-1)(2m-1)(2m-2+\lambda)(2m-1+\lambda)[m^2 + (\lambda-2)m - \lambda + \frac{1}{2}]}{2(2\lambda+1)}. \quad (3.12)$$

The solution of (3.12) with the initial condition $\tilde{D}_{2,1} = 0$ is given by

$$\tilde{D}_{2,m} = \frac{(m-1)(2m-1)(m+\lambda)(2m-1+\lambda)(2m-1+2\lambda)[m^2 + (\lambda-1)m - \frac{2\lambda+1}{2} - \frac{2}{2\lambda+3}]}{2(2\lambda+1)(2\lambda+5)}. \quad (3.13)$$

Proof. The recurrence formula (3.10) is deduced by two-fold differentiation of (3.4) with respect to μ , then setting $\mu = 0$ and using Lemma 3.1(i) to replace $A_{1,m-1}$ in the resulting identity. The recurrence formula (3.12) is obtained in the same manner: we differentiate (3.5) twice, then set $\mu = 0$ and apply Lemma 3.1(ii) to replace $\tilde{A}_{1,m-1}$ in the resulting identity.

Now it is a straightforward (though rather tedious) task to verify that the sequences $\{D_{2,m}\}$ and $\{\tilde{D}_{2,m}\}$ defined by (3.11) and (3.13) are the solutions of the recurrence relations (3.10) and (3.12), respectively, with the initial conditions $D_{2,1} = 0$, $\tilde{D}_{2,1} = 0$. \square

Lemma 3.4 *The coefficients $A_{2,m}$ and $\tilde{A}_{2,m}$ are given by*

$$A_{2,m} = \frac{(m-1)m(m+1)(m+\lambda)(m+\lambda+1)(m+\lambda+2)\left[m^2 + (\lambda+1)m + \frac{4\lambda^2+2\lambda-14}{3(2\lambda+3)}\right]}{2(2\lambda+1)(2\lambda+5)} \quad (3.14)$$

and

$$\tilde{A}_{2,m} = \frac{(m-1)m(m+\lambda)(m+\lambda+1)r_\lambda(m)}{24(2\lambda+1)(2\lambda+3)(2\lambda+5)}, \quad (3.15)$$

$$r_\lambda(m) := 12(2\lambda+3)m^4 + 24\lambda(2\lambda+3)m^3 + 4(6\lambda^3 + 7\lambda^2 - 19\lambda - 32)m^2 - 4\lambda(2\lambda^2 + 19\lambda + 32)m - 8\lambda^3 - 20\lambda^2 + 14\lambda + 71. \quad (3.16)$$

Proof. We have

$$A_{2,m} = \sum_{k=2}^m D_{2,k}, \quad \tilde{A}_{2,m} = \sum_{k=2}^m \tilde{D}_{2,k},$$

hence, knowing formulae (3.14) and (3.15)–(3.16), one may think of proving them by induction with respect to m , especially having in mind that the induction base is obvious. However, performing the induction step by hand, though possible, is a hard work, this is why we highly recommend for that purpose the usage of a computer algebra program, for instance, Wolfram's Mathematica does perfectly that job.

A reasonable question here is: how do we guess formulae (3.14) and (3.15)–(3.16)? Our approach makes use of the observation that $A_{2,m}$ and $\tilde{A}_{2,m}$ are polynomials in m . We evaluate these coefficients for several consecutive values of m (nine values suffice!) and then construct the associated interpolating polynomials to deduce the expressions for $A_{2,m}$ and $\tilde{A}_{2,m}$. Needless to say, we have used a computer algebra program for this purpose. \square

Next, we obtain two-sided estimates for the coefficients $A_{2,m}$ and $\tilde{A}_{2,m}$.

Lemma 3.5 *For all $m \in \mathbb{N}$, $m \geq 2$, and for every $\lambda > -\frac{1}{2}$, the coefficient $A_{2,m}$ admits the estimates*

$$\frac{(m-1)m^2(m+1)(m+\lambda)^2(m+\lambda+1)^2}{2(2\lambda+1)(2\lambda+5)} \leq A_{2,m} \leq \frac{(m-1)m(m+1)^2(m+\lambda)^2(m+\lambda+1)(m+\lambda+2)}{2(2\lambda+1)(2\lambda+5)}.$$

Proof. We use formula (3.14). For the lower estimate, we need to show that

$$(m+\lambda+2)\left[m^2 + (\lambda+1)m + \frac{4\lambda^2+2\lambda-14}{3(2\lambda+3)}\right] \geq m(m+\lambda)(m+\lambda+1).$$

The difference of the left-hand and the right-hand sides is equal to

$$g_0(m) := 2m^2 + \frac{4(4\lambda^2+8\lambda+1)}{3(2\lambda+3)}m + \frac{2(\lambda+2)(2\lambda^2+\lambda-7)}{3(2\lambda+3)}.$$

It is easy to see that $g'_0(m) > 0$ for $m \geq 2$ and $\lambda > -1/2$, therefore g_0 is monotone increasing, and

$$g_0(m) \geq g_0(2) = \frac{2(2\lambda^3+21\lambda^2+51\lambda+26)}{3(2\lambda+3)} > 0.$$

For the upper estimate, we need to prove the inequality

$$m^2 + (\lambda + 1)m + \frac{4\lambda^2 + 2\lambda - 14}{3(2\lambda + 3)} \leq (m + 1)(m + \lambda).$$

The latter is equivalent to the inequality

$$\frac{4\lambda^2 + 2\lambda - 14}{3(2\lambda + 3)} < \lambda,$$

which is readily verified to be true for $\lambda > -1/2$. \square

Lemma 3.6 For all $m \in \mathbb{N}$, $m \geq 2$, the coefficient $\tilde{A}_{2,m}$ admits the lower estimates

$$(i) \quad \tilde{A}_{2,m} \geq \frac{(m-1)m(m+\lambda)(m+\lambda+1)(m^2 + \lambda m - \frac{1}{2})(m^2 + \lambda m - \frac{\lambda}{3} - \frac{7}{2})}{2(2\lambda+1)(2\lambda+5)}, \quad -\frac{1}{2} < \lambda \leq 0,$$

$$(ii) \quad \tilde{A}_{2,m} \geq \frac{(m-1)m(m+\lambda)(m+\lambda+1)(m^2 + \lambda m - \frac{1}{2})(m^2 + \lambda m - \frac{\lambda}{2} - \frac{7}{2})}{2(2\lambda+1)(2\lambda+5)}, \quad \lambda \geq 0.$$

For all $m \in \mathbb{N}$, $m \geq 2$, and for every $\lambda > -\frac{1}{2}$, the coefficient $\tilde{A}_{2,m}$ admits the upper estimate

$$\tilde{A}_{2,m} \leq \frac{(m-1)m^2(m+\lambda)^2(m+\lambda+1)(m^2 + \lambda m - \frac{1}{2})}{2(2\lambda+1)(2\lambda+5)}.$$

Proof. The polynomial r_λ in (3.16) satisfies

$$r_\lambda(m) = (2\lambda+3)(12m^4 + 24\lambda m^3 + (12\lambda^2 - 4\lambda - 32)m^2 - (4\lambda^2 + 32\lambda + 16)m - 4\lambda^2 - 4\lambda + 13) - 16(m-2)(2m+1), \quad (3.17)$$

therefore

$$r_\lambda(m) \leq (2\lambda+3)(12m^4 + 24\lambda m^3 + (12\lambda^2 - 4\lambda - 32)m^2 - (4\lambda^2 + 32\lambda + 16)m - 4\lambda^2 - 4\lambda + 13) =: (2\lambda+3)s_\lambda(m).$$

On the other hand,

$$s_\lambda(m) = (12m^2 + 12\lambda m - 4\lambda - 26)(m^2 + \lambda m - \frac{1}{2}) - (16m + 4\lambda^2 + 6\lambda) < 12m(m+\lambda)(m^2 + \lambda m - \frac{1}{2}),$$

hence

$$r_\lambda(m) \leq 12(2\lambda+3)m(m+\lambda)(m^2 + \lambda m - \frac{1}{2}).$$

The upper estimate for $\tilde{A}_{2,m}$ now follows by putting this upper bound for r_λ in (3.15).

For the proof of the lower estimates for $\tilde{A}_{2,m}$, we estimate from below the factor r_λ in (3.15). Since $-16 > -8(2\lambda+3)$, replacement of -16 by $-8(2\lambda+3)$ in the second line of (3.17) yields

$$r_\lambda(m) \geq (2\lambda+3)(12m^4 + 24\lambda m^3 + (12\lambda^2 - 4\lambda - 48)m^2 - (4\lambda^2 + 32\lambda - 8)m - 4\lambda^2 - 4\lambda + 29) =: (2\lambda+3)\tilde{s}_\lambda(m).$$

Next, we estimate \tilde{s}_λ from below, distinguishing between the cases $-\frac{1}{2} < \lambda \leq 0$ and $\lambda \geq 0$. If $-\frac{1}{2} < \lambda \leq 0$, then from

$$\begin{aligned} \tilde{s}_\lambda(m) &= 12(m^2 + \lambda m - \frac{1}{2})(m^2 + \lambda m - \frac{\lambda}{3} - \frac{7}{2}) + 8(2\lambda+1)m - 4\lambda^2 - 6\lambda + 8 \\ &> 12(m^2 + \lambda m - \frac{1}{2})(m^2 + \lambda m - \frac{\lambda}{3} - \frac{7}{2}), \quad -\frac{1}{2} < \lambda \leq 0 \end{aligned}$$

(clearly, in that case $8(2\lambda + 1)m - 4\lambda^2 - 6\lambda + 8 > 0$) we deduce the lower bound (i).

If $\lambda \geq 0$, then the lower bound (ii) follows from

$$\begin{aligned}\tilde{s}_\lambda(m) &= 12\left(m^2 + \lambda m - \frac{1}{2}\right)\left(m^2 + \lambda m - \frac{\lambda}{2} - \frac{7}{2}\right) + 2\lambda m^2 + (2\lambda^2 + 16\lambda + 8)m - 4\lambda^2 - 7\lambda + 8 \\ &> 12\left(m^2 + \lambda m - \frac{1}{2}\right)\left(m^2 + \lambda m - \frac{\lambda}{2} - \frac{7}{2}\right), \quad \lambda \geq 0, \quad m \geq 2.\end{aligned}$$

For the last inequality we have used that, for $\lambda \geq 0$ and $m \geq 2$,

$$g_1(m) := 2\lambda m^2 + (2\lambda^2 + 16\lambda + 8)m - 4\lambda^2 - 7\lambda + 8 \geq g_1(2) = 33\lambda + 24 > 0.$$

Lemma 3.6 is proved. \square

4 Estimates for the best Markov constant $c_n(\lambda)$

Let us recall that Q_m and \tilde{Q}_m are the characteristic polynomials of the matrices $\mathbf{B}_m = \mathbf{A}_m^{-1}$ and $\tilde{\mathbf{B}}_m = \tilde{\mathbf{A}}_m^{-1}$, respectively, normalized by $Q_m(0) = \tilde{Q}_m(0) = 1$. Hence, their reciprocal polynomials,

$$R_m(x) = x^m Q_m(x^{-1}) = x^m - A_{1,m}x^{m-1} + A_{2,m}x^{m-2} - \cdots + (-1)^m A_{m,m}, \quad (4.1)$$

$$\tilde{R}_m(x) = x^m \tilde{Q}_m(x^{-1}) = x^m - \tilde{A}_{1,m}x^{m-1} + \tilde{A}_{2,m}x^{m-2} - \cdots + (-1)^m \tilde{A}_{m,m}, \quad (4.2)$$

are the monic characteristic polynomials of matrices \mathbf{A}_m and $\tilde{\mathbf{A}}_m$, respectively. In Sect. 2 we showed that Q_m and \tilde{Q}_m are polynomials orthogonal with respect to measures supported on the positive axis, therefore their zeros are single and positive. Then the same observation applies to the zeros of R_m and \tilde{R}_m , which we denote by $\{\nu_i\}$ and $\{\tilde{\nu}_i\}$, respectively, so that

$$\begin{aligned}R_m(x) &= (x - \nu_1)(x - \nu_2) \cdots (x - \nu_m), & 0 < \nu_1 < \nu_2 < \cdots < \nu_m, \\ \tilde{R}_m(x) &= (x - \tilde{\nu}_1)(x - \tilde{\nu}_2) \cdots (x - \tilde{\nu}_m), & 0 < \tilde{\nu}_1 < \tilde{\nu}_2 < \cdots < \tilde{\nu}_m.\end{aligned}$$

Our tool for obtaining two-sided estimates for ν_m and $\tilde{\nu}_m$ is the following simple observation:

Proposition 4.1 *Let*

$$f(x) = x^m - a_{1,m}x^{m-1} + a_{2,m}x^{m-2} - \cdots + (-1)^m a_{m,m}$$

be a polynomial having only real and positive zeros $\{x_i\}$, $0 < x_1 \leq x_2 \leq \cdots \leq x_m$. Then

$$a_{1,m} - 2 \frac{a_{2,m}}{a_{1,m}} \leq x_m \leq \sqrt{a_{1,m}^2 - 2a_{2,m}}.$$

In either place, the equality holds if and only if $x_1 = x_2 = \cdots = x_m$.

Proof. The claim is equivalent to

$$\frac{x_1^2 + x_2^2 + \cdots + x_m^2}{x_1 + x_2 + \cdots + x_m} \leq x_m \leq (x_1^2 + x_2^2 + \cdots + x_m^2)^{\frac{1}{2}},$$

and both the inequalities and the equality cases are obvious. \square

We obtain separately estimates for $c_n(\lambda)$ for even and odd n . Theorem 1.1 is then obtained as a summary of these results.

4.1 The cases of even and odd n

According to Theorem 2.1, for the best Markov constant $c_n(\lambda)$ we have

$$c_{2m}^2(\lambda) = 4\nu_m, \quad (4.3)$$

$$c_{2m-1}^2(\lambda) = 4\tilde{\nu}_m. \quad (4.4)$$

Theorem 4.2 For all even n , $n \geq 4$, and for every $\lambda > -\frac{1}{2}$ the best Markov constant $c_n(\lambda)$ admits the estimates

$$\frac{(n+2)(n+2\lambda)(n+\lambda+\frac{1}{2})^2}{(2\lambda+1)(2\lambda+5)} \leq c_n(\lambda)^2 \leq \frac{n(n+2\lambda)(n+2\lambda+2)\sqrt{(n+2)(n+2\lambda+3)}}{2(2\lambda+1)\sqrt{2\lambda+5}}.$$

Proof. Let us set $n = 2m$. We apply Proposition 4.1 with $f = R_m$, making use of Lemma 3.1(i) and Lemma 3.5.

1) To derive the lower bound for $c_n(\lambda)^2$, we estimate

$$\begin{aligned} \nu_m &\geq A_{1,m} - 2 \frac{A_{2,m}}{A_{1,m}} \geq \frac{m(m+1)(m+\lambda)(m+\lambda+1)}{2\lambda+1} - \frac{(m-1)(m+1)(m+\lambda)(m+\lambda+2)}{2\lambda+5} \\ &= \frac{(m+1)(m+\lambda)}{(2\lambda+1)(2\lambda+5)} [4m(m+\lambda+1) + (2\lambda+1)(\lambda+2)] \\ &\geq \frac{(m+1)(m+\lambda)(2m+\lambda+\frac{1}{2})^2}{(2\lambda+1)(2\lambda+5)}. \end{aligned}$$

Hence,

$$c_n^2(\lambda) = 4\nu_m \geq \frac{4(m+1)(m+\lambda)(2m+\lambda+\frac{1}{2})^2}{(2\lambda+1)(2\lambda+5)} = \frac{(n+2)(n+2\lambda)(n+\lambda+\frac{1}{2})^2}{(2\lambda+1)(2\lambda+5)}.$$

2) For the upper estimate in Theorem 4.2, we have

$$\begin{aligned} \nu_m^2 &\leq A_{1,m}^2 - 2A_{2,m} \leq \frac{m^2(m+1)^2(m+\lambda)^2(m+\lambda+1)^2}{(2\lambda+1)^2} - \frac{(m-1)m^2(m+1)(m+\lambda)^2(m+\lambda+1)^2}{(2\lambda+1)(2\lambda+5)} \\ &= \frac{4m^2(m+\lambda)^2(m+\lambda+1)^2(m+1)(m+\lambda+\frac{3}{2})}{(2\lambda+1)^2(2\lambda+5)} = \frac{n^2(n+2)(n+2\lambda)^2(n+2\lambda+2)(n+2\lambda+3)}{64(2\lambda+1)^2(2\lambda+5)} \end{aligned}$$

and then (4.3) yields

$$c_n^2(\lambda) = 4\nu_m \leq \frac{n(n+2\lambda)(n+2\lambda+2)\sqrt{(n+2)(n+2\lambda+3)}}{2(2\lambda+1)\sqrt{2\lambda+5}}.$$

The proof of Theorem 4.2 is complete. \square

Remark 4.3 For $\lambda \geq 2$ the upper bound for $c_n^2(\lambda)$ in Theorem 4.2 admits a slight improvement, namely, we have

$$c_n^2(\lambda) \leq \frac{n(n+2\lambda)(n+2\lambda+2)\sqrt{(n+2)(n+2\lambda+2)}}{2(2\lambda+1)\sqrt{2\lambda+5}}, \quad \lambda \geq 2. \quad (4.5)$$

Indeed, for $\lambda \geq 2$ we can replace the lower bound for $A_{2,m}$ in Lemma 3.5 by the sharper one

$$A_{2,m} \geq \frac{(m-1)m^2(m+1)(m+\lambda)(m+\lambda+1)^2(m+\lambda+2)}{2(2\lambda+1)(2\lambda+5)},$$

and then, proceeding in the same way as above, we arrive at the estimate (4.5)

Theorem 4.4 For all odd n , $n \geq 3$, and for every $\lambda > -\frac{1}{2}$, the best Markov constant $c_n(\lambda)$ admits the estimates

$$\frac{(n+1)(n+\lambda+\frac{1}{2})^2(n+2\lambda+1)}{(2\lambda+1)(2\lambda+5)} \leq c_n^2(\lambda) \leq \frac{(n+1)^{\frac{3}{2}}(n+2\lambda+1)^2(n+2\lambda'+1)^{\frac{1}{2}}}{2(2\lambda+1)\sqrt{2\lambda+5}},$$

where $\lambda' = \max\{\lambda, 0\}$.

Proof. Let us set $n = 2m - 1$, $m \geq 2$. We apply Proposition 4.1 with $f = \tilde{R}_m$, making use of Lemma 3.1(ii) and Lemma 3.6.

1) For the lower bound, we estimate $\tilde{\nu}_m$ from below, using Proposition 4.1, Lemma 3.1(ii) and Lemma 3.6 to obtain

$$\begin{aligned} \tilde{\nu}_m &\geq \tilde{A}_{1,m} - 2 \frac{\tilde{A}_{2,m}}{\tilde{A}_{1,m}} \geq \frac{m(m+\lambda)(m^2+\lambda m - \frac{1}{2})}{2\lambda+1} - \frac{(m-1)m(m+\lambda)(m+\lambda+1)}{2\lambda+5} \\ &= \frac{m(m+\lambda)}{(2\lambda+1)(2\lambda+5)} \left[4m^2 + 4\lambda m + 2\lambda^2 + 2\lambda - \frac{3}{2} \right] \geq \frac{m(m+\lambda)(2m+\lambda - \frac{1}{2})^2}{(2\lambda+1)(2\lambda+5)}. \end{aligned}$$

Now the lower estimate for $c_n^2(\lambda)$ follows from (4.4):

$$c_n^2(\lambda) = 4\tilde{\nu}_m \geq \frac{2m(2m+2\lambda)(2m+\lambda - \frac{1}{2})^2}{(2\lambda+1)(2\lambda+5)} = \frac{(n+1)(n+2\lambda+1)(n+\lambda + \frac{1}{2})^2}{(2\lambda+1)(2\lambda+5)}.$$

2) Next, we prove the upper estimate for $c_n^2(\lambda)$.

2.1) In the case $-\frac{1}{2} < \lambda \leq 0$, we apply Proposition 4.1, Lemma 3.1(ii) and inequality (i) in Lemma 3.6 to estimate $\tilde{\nu}_m^2$ from above as follows:

$$\begin{aligned} \tilde{\nu}_m^2 &\leq \tilde{A}_{1,m}^2 - 2\tilde{A}_{2,m} \leq \frac{m^2(m+\lambda)^2(m^2+\lambda m - \frac{1}{2})^2}{(2\lambda+1)^2} - \frac{(m-1)m(m+\lambda)(m+\lambda+1)(m^2+\lambda m - \frac{1}{2})(m^2+\lambda m - \frac{\lambda}{3} - \frac{7}{2})}{(2\lambda+1)(2\lambda+5)} \\ &= \frac{m(m+\lambda)(m^2+\lambda m - \frac{1}{2})}{(2\lambda+1)^2(2\lambda+5)} \left[(2\lambda+5)m(m+\lambda)(m^2+\lambda m - \frac{1}{2}) - (2\lambda+1)(m-1)(m+\lambda+1)(m^2+\lambda m - \frac{\lambda}{3} - \frac{7}{2}) \right] \\ &= \frac{4m(m+\lambda)(m^2+\lambda m - \frac{1}{2})}{(2\lambda+1)^2(2\lambda+5)} \left[m^2(m+\lambda)^2 + \left(\frac{2}{3}\lambda^2 + \frac{7}{3}\lambda + \frac{1}{2} \right) m(m+\lambda) - \frac{1}{4}(2\lambda+1)(\lambda+1) \left(\frac{\lambda}{3} + \frac{7}{2} \right) \right] \\ &\leq \frac{4m^2(m+\lambda)^2(m^2+\lambda m - \frac{1}{2})}{(2\lambda+1)^2(2\lambda+5)} \left[m(m+\lambda) + \frac{2}{3}\lambda^2 + \frac{7}{3}\lambda + \frac{1}{2} \right]. \end{aligned}$$

Since $g_2(\lambda) := \frac{2}{3}\lambda^2 + \frac{7}{3}\lambda + \frac{1}{2}$ is a monotone increasing function in $(-1/2, 0]$, the expression in the last brackets does not exceed $m^2 + \lambda m + \frac{1}{2}$, hence

$$\tilde{\nu}_m^2 \leq \frac{4m^2(m+\lambda)^2[m^2(m+\lambda)^2 - \frac{1}{4}]}{(2\lambda+1)^2(2\lambda+5)} \leq \frac{4m^4(m+\lambda)^4}{(2\lambda+1)^2(2\lambda+5)}.$$

Now from (4.4) we obtain the desired upper estimate for $c_n^2(\lambda)$:

$$c_n^2(\lambda) = 4\tilde{\nu}_m \leq \frac{8m^2(m+\lambda)^2}{(2\lambda+1)^2\sqrt{2\lambda+5}} = \frac{(n+1)^2(n+2\lambda+1)^2}{2(2\lambda+1)\sqrt{2\lambda+5}} = \frac{(n+1)^{\frac{3}{2}}(n+2\lambda'+1)^{\frac{1}{2}}(n+2\lambda+1)^2}{(2\lambda+1)\sqrt{2\lambda+5}}.$$

2.2) In view of (4.4), in the case $\lambda \geq 0$ the upper estimate for $c_n^2(\lambda)$ in Theorem 4.4 is equivalent to

$$\tilde{\nu}_m^2 \leq \frac{4m^3(m+\lambda)^5}{(2\lambda+1)^2(2\lambda+5)}. \quad (4.6)$$

We apply Proposition 4.1, Lemma 3.1(ii) and inequality (ii) in Lemma 3.6 to estimate $\tilde{\nu}_m^2$ from above as follows:

$$\begin{aligned}\tilde{\nu}_m^2 &\leq \frac{m^2(m+\lambda)^2(m^2+\lambda m-\frac{1}{2})^2}{(2\lambda+1)^2} - \frac{(m-1)m(m+\lambda)(m+\lambda+1)(m^2+\lambda m-\frac{1}{2})(m^2+\lambda m-\frac{\lambda+7}{2})}{(2\lambda+1)(2\lambda+5)} \\ &= \frac{m(m+\lambda)(m^2+\lambda m-\frac{1}{2})}{(2\lambda+1)^2(2\lambda+5)} \left[(2\lambda+5)(m^2+\lambda m)(m^2+\lambda m-\frac{1}{2}) - (2\lambda+1)(m^2+\lambda m-\lambda-1)(m^2+\lambda m-\frac{\lambda+7}{2}) \right] \\ &= \frac{m(m+\lambda)(m^2+\lambda m-\frac{1}{2})}{(2\lambda+1)^2(2\lambda+5)} \left[4m^2(m+\lambda)^2 + \frac{1}{2}(6\lambda^2+19\lambda+4)m(m+\lambda) - \frac{1}{2}(2\lambda+1)(\lambda+1)(\lambda+7) \right] \\ &\leq \frac{4m^2(m+\lambda)^2}{(2\lambda+1)^2(2\lambda+5)} (m^2+\lambda m-\frac{1}{2}) \left[m^2+\lambda m + \frac{1}{8}(6\lambda^2+19\lambda+4) \right].\end{aligned}$$

To prove (4.6), it suffices to show that

$$(m^2+\lambda m-\frac{1}{2}) \left[m^2+\lambda m + \frac{1}{8}(6\lambda^2+19\lambda+4) \right] \leq m(m+\lambda)^3, \quad \lambda \geq 0, \quad m \geq 2.$$

For $m \geq 3$ the above inequality follows from

$$m(m+\lambda)^3 - (m^2+\lambda m-\frac{1}{2}) \left[m^2+\lambda m + \frac{1}{8}(6\lambda^2+19\lambda+4) \right] = \frac{1}{8} \lambda m(m+\lambda)(8m+2\lambda-19) + \frac{1}{16}(6\lambda^2+19\lambda+4),$$

while for $m = 2$ it is equivalent to the inequality

$$8\lambda^3 + 10\lambda^2 - 5\lambda + 4 \geq 0, \quad \lambda \geq 0,$$

which is readily verified to be true. \square

4.2 Proof of Theorem 1.1 and Corollary 1.3

Proof of Theorem 1.1. Clearly, the lower bound for $c_n^2(\lambda)$ in Theorem 1.1 is smaller than the lower bounds in Theorems 4.2 and 4.4, hence it is a lower bound in the cases of both even and odd n .

Next, we prove the upper bound for $c_n^2(\lambda)$ in Theorem 1.1. To this end, we apply the geometric mean - arithmetic mean inequality in Theorems 4.2 and 4.4 to obtain

$$\begin{aligned}c_n^2(\lambda) &\leq \frac{\left(n+\frac{5}{4}\lambda+\frac{9}{8}\right)^4}{2(2\lambda+1)\sqrt{2\lambda+5}}, \quad n = 2m, \\ c_n^2(\lambda) &\leq \frac{\left(n+\lambda+\frac{\lambda'}{4}+1\right)^4}{2(2\lambda+1)\sqrt{2\lambda+5}}, \quad n = 2m-1, \quad \lambda' = \max\{0, \lambda\}\end{aligned}$$

and compare the right-hand sides of these inequalities, observing that the first one is the greater. \square

Remark 4.5 Applying the geometric mean - arithmetic mean inequality to the upper bounds for $c_n^2(\lambda)$ in Theorems 4.2 and 4.4 to obtain the upper bound in Theorem 1.1, we certainly lose. For instance, for a fixed n , the upper bounds in Theorems 4.2 and 4.4 are $O(\lambda)$ as $\lambda \rightarrow \infty$ (notice that the same applies to the lower bounds therein!), while the resulting upper bound in Theorem 1.1 is $O(\lambda^{\frac{5}{2}})$ as $\lambda \rightarrow \infty$. However, as was already said, the upper estimates here are good for relatively small λ , say, $\lambda \leq 25$. For big λ , we have the better upper estimates (1.2) in Theorem A.

Proof of Corollary 1.3. The comparison of the bounds for $c_n^2(\lambda)$ in Theorems 4.2 and 4.4 reveals that for $\lambda < \frac{1}{2}$ the smaller lower bound is the one in Theorem 4.2, while in the limit case $\lambda = -1/2$ the bigger numerator has the upper bound in Theorem 4.4. By taking the limits in the expressions obtained from corresponding bounds we obtain the result. \square

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